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On some refinement of the Cauchy–Schwarz inequality

Shuhei Wada

Department of Information and Computer Engineering, Kisarazu National College of Technology, 2-11-1 Kiyomidai-Higashi, Kisarazu, Chiba 292-0041, Japan

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Abstract

If A and B are positive semidefinite operators on a Hilbert space and if σ is an operator mean in the sense of Kubo and Ando, then the operator inequality

$$(A\#B)\otimes(A\#B)\leqslant\frac{1}{2}\left\{(A\sigma B)\otimes(A\sigma^{\perp}B)+(A\sigma^{\perp}B)\otimes(A\sigma B)\right\}$$
$$\leqslant\frac{1}{2}\{(A\otimes B)+(B\otimes A)\}$$

holds. This inequality is a generalization of some refinement of the Cauchy–Schwarz inequality.

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1. Introduction

Several refinements of the Cauchy–Schwarz inequality have been studied by many authors. Among them, the result due to Daykin–Eliezer–Carlitz [5] attracts our interest. They characterize the pair of functions (f, g) which satisfies

$$\left\{\sum_{1}^{n}\sqrt{a_i b_i}\right\}^2\leqslant\sum_{1}^{n}f(a_i,b_i)\sum_{1}^{n}g(a_i,b_i)\leqslant\sum_{1}^{n}a_i\sum_{1}^{n}b_i$$

E-mail address: wada@j.kisarazu.ac.jp

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for any positive real numbers $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$. This inequality comes out by taking trace of the matrix inequality of the following type:

$$\begin{aligned}(A\#B)\otimes(A\#B) &\leqslant \frac{1}{2}\{F(A,B)\otimes G(A,B)+G(A,B)\otimes F(A,B)\}\\ &\leqslant \frac{1}{2}\{(A\otimes B)+(B\otimes A)\},\end{aligned}$$

where F and G are binary operations of square matrices and A, B are diagonal positive matrices and # is the geometric mean. From this viewpoint, we study the matrix inequality mentioned just above. We come to get the fact that this inequality implies some well-known numerical inequalities if F and G is an operator mean and its dual, respectively.

The main purpose of this paper is to give an operator generalization of the above matrix inequality.

Theorem 1. *Let A and B be positive semidefinite operators on a Hilbert space. Then*

$$\begin{aligned}(A\#B)\otimes(A\#B) &\leqslant \frac{1}{2}\left\{(A\sigma B)\otimes(A\sigma^{\perp}B)+(A\sigma^{\perp}B)\otimes(A\sigma B)\right\}\\ &\leqslant \frac{1}{2}\{(A\otimes B)+(B\otimes A)\},\end{aligned} \tag{1}$$

where σ and $\sigma^{\perp}$ are an operator mean and its dual in the sense of Kubo and Ando.

2. Preliminaries

In this paper, a binary operation for a pair of positive semidefinite operators on a Hilbert space, called an operator mean, plays an important role. The purpose of this section is to recall some well-known results concerning operator means.

Let A, B, C, D be bounded linear operators on a Hilbert space. The order relation $A \leqslant B$ is defined by $B - A \geqslant 0$ (i.e. $B - A$ is positive semidefinite). If $A - B$ is positive definite, then we denote by $A > B$. The fact that positive semidefinite operators $\{A_n\}$ converges strongly to A and $A_1 \geqslant A_2 \geqslant \cdots$ is denoted by $A_n \downarrow A$.

The following axiom of operator means is given by Kubo and Ando (see [6,7]). A binary operation σ among the cone $\mathscr{P}(\mathscr{H})$ of positive semidefinite operators on a Hilbert space $\mathscr{H}$ is an operator mean if it satisfies the following:

(i) $A \leqslant C, B \leqslant D \Rightarrow A\sigma B \leqslant C\sigma D$,
(ii) $T^*(A\sigma B)T \leqslant (T^*AT)\sigma(T^*BT)$,
(iii) $A_n \downarrow A, B_n \downarrow B \Rightarrow A_n\sigma B_n \downarrow A\sigma B$.

A real valued continuous function ξ on $(0, \infty)$ is called operator monotone if $\xi(A) \leqslant \xi(B)$ whenever $A \leqslant B$ and $A > 0, B > 0$. It is well-known that an operator monotone function is a concave function (see [2]). If ξ is a non-negative operator monotone function on $(0, \infty)$, then the binary operation σ_ξ on $\mathscr{P}(\mathscr{H})$ defined by

$$A\sigma_\xi B = \lim_{\epsilon\downarrow 0} A_\epsilon^{\frac{1}{2}}\xi\left(A_\epsilon^{-\frac{1}{2}}B_\epsilon^{\frac{1}{2}}A_\epsilon^{-\frac{1}{2}}\right)A_\epsilon^{\frac{1}{2}}$$

is an operator mean, where $A_\epsilon = A + \epsilon 1$ and $B_\epsilon = B + \epsilon 1$. The definition of the map $\xi \mapsto \sigma_\xi$ implies the following property:

$$\sigma_{\alpha\xi_1+\beta\xi_2} = \alpha\sigma_{\xi_1} + \beta\sigma_{\xi_2} \quad (\alpha, \beta \geqslant 0).$$

The map satisfying this property is called an affine homomorphism. In addition to this, the map $\xi \mapsto \sigma_\xi$ has an important property of order preserving, that is, if ξ_1 and ξ_2 are operator monotone functions satisfying $\xi_1(t) \leqslant \xi_2(t)$, then

$$A\sigma_{\xi_1} B \leqslant A\sigma_{\xi_2} B \quad (A, B \geqslant 0).$$

Proposition 2. *The map $\xi \mapsto \sigma_\xi$ establishes an order-preserving affine isomorphism from the class of non-negative operator monotone functions on* $(0, \infty)$ *onto the class of operator means.*

Remark. It is clear that the inverse of the map $\xi \mapsto \sigma_\xi$ is a map from an operator mean σ to an operator monotone function ξ_σ and satisfies $\xi_\sigma(t)1 = 1\sigma(t1)$. The operator monotone function ξ_σ associated with σ is called the representing function of σ.

From the preceding proposition, it follows that every operator mean can be defined by using an operator monotone function. For example, the arithmetic mean $\triangledown$ and the geometric mean # are defined by

$$\frac{1+t}{2} \quad \text{and} \quad \sqrt{t},$$

respectively. By the fact that the map $\xi \mapsto \sigma_\xi$ has the property of order preserving, the following operator arithmetic–geometric inequality can be proved:

$$A\#B \leqslant A\triangledown B \quad (A, B \geqslant 0).$$

Note that $A\triangledown B$ can be written as $\frac{A+B}{2}$.

From the preceding proposition, it is clear that every map on the class of operator means can be defined by using a map on the class of non-negative operator monotone functions. For example, let ξ be the operator monotone function associated with σ. The operator mean $\sigma^\perp$ which is called the dual of σ satisfies $A\sigma^\perp B = (B^{-1}\sigma A^{-1})^{-1}$ for $A > 0$, $B > 0$. The representing function η of $\sigma^\perp$ satisfies $\eta(t) = \frac{t}{\xi(t)}$.

3. Refinement

In this section, we first state a refinement of the discrete Cauchy–Schwarz inequality obtained by Daykin–Eliezer–Carlitz [5].

Proposition 3 (Daykin–Eliezer–Carlitz). *Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be positive real numbers and let $f(\cdot,\cdot)$ and $g(\cdot,\cdot)$ be positive functions with two variables. The inequality*

$$\left\{\sum_1^n \sqrt{a_i b_i}\right\}^2 \leqslant \sum_1^n f(a_i, b_i) \sum_1^n g(a_i, b_i) \leqslant \sum_1^n a_i \sum_1^n b_i, \tag{2}$$

holds if and only if

(i) $f(a, b)g(a, b) = ab$,
(ii) $f(ka, kb) = kf(a, b)$ $(k > 0)$,
(iii) $f(1, b) \leqslant f(1, a)$, $\frac{f(1,a)}{a} \leqslant \frac{f(1,b)}{b}$ $(b \leqslant a)$.

It is easily verified that the preceding proposition also holds if the third condition is replaced by

$$f(a_1, b_1) \leqslant f(a_2, b_2), \quad (a_1 \leqslant a_2, b_1 \leqslant b_2).$$

In the case that f and g are the arithmetic mean and its dual, that is,

$$f(a, b) = \frac{a+b}{2} \quad \text{and} \quad g(a, b) = \frac{2ab}{a+b},$$

the derived inequality (2) is described as follows.

Proposition 4. *Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be non-negative real numbers. Then the following inequality holds*:

$$\left\{\sum_1^n \sqrt{a_i b_i}\right\}^2 \leqslant \sum_1^n (a_i + b_i) \sum_1^n \frac{a_i b_i}{a_i + b_i} \leqslant \sum_1^n a_i \sum_1^n b_i.$$

This result has been called the Milne's inequality [8].

If f and g are t-power mean and $(1-t)$-power mean, that is

$$f(a, b) = a^{1-t} b^t \quad \text{and} \quad g(a, b) = a^t b^{1-t},$$

then the following inequality has been called the Callebaut inequality [3].

Proposition 5. *Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be non-negative real numbers. Then the following inequality holds*:

$$\left\{\sum_1^n \sqrt{a_i b_i}\right\}^2 \leqslant \sum_1^n a_i^{1-t} b_i^t \sum_1^n a_i^t b_i^{1-t} \leqslant \sum_1^n a_i \sum_1^n b_i,$$

where $t \in [0, 1]$.

In [3], Callebaut also proves the monotonicity of the function

$$t \mapsto \sum_1^n a_i^{1-t} b_i^t \sum_1^n a_i^t b_i^{1-t}.$$

Proposition 6. *Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be non-negative real numbers. If $1 \geqslant t \geqslant s \geqslant \frac{1}{2}$ or $\frac{1}{2} \geqslant s \geqslant t \geqslant 0$, then*

$$\sum_1^n a_i^{1-t} b_i^t \sum_1^n a_i^t b_i^{1-t} \geqslant \sum_1^n a_i^{1-s} b_i^s \sum_1^n a_i^s b_i^{1-s}.$$

Particularly, in the case when $n = 2$, $a_1 = b_2$ and $a_2 = b_1$, the following result is obtained.

Corollary 7. *Let a and b be positive numbers. If $1 \geqslant t \geqslant s \geqslant \frac{1}{2}$ or $\frac{1}{2} \geqslant s \geqslant t \geqslant 0$, then*

$$a^{1-t} b^t + a^t b^{1-t} \geqslant a^{1-s} b^s + a^s b^{1-s}.$$

4. Operator inequality

4.1. Cauchy–Schwarz inequality

Let $\mathscr{H}$ be a Hilbert space and let A and B be bounded linear operators on $\mathscr{H}$. The operator $A \otimes B$ on $\mathscr{H} \otimes \mathscr{H}$ are defined by

$$(A \otimes B)(x \otimes y) = Ax \otimes By$$

for every $x, y \in \mathscr{H}$ and extend by linearity to $\mathscr{H} \otimes \mathscr{H}$ (see [2]). From this definition, it is clear that the tensor product of positive semidefinite operators is positive semidefinite. Let A and B be two matrices. If $A = (a_{ij})$, then $A \otimes B$ can be written as

$$A \otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \cdots & \cdots & \cdots \\ a_{n1}B & \cdots & a_{nn}B \end{pmatrix}.$$

In the inequality (1) of Theorem 1, the part

$$(A\#B) \otimes (A\#B) \leqslant \frac{1}{2}\{(A \otimes B) + (B \otimes A)\} \tag{3}$$

is easily seen as follows. By the definitions of # and $\otimes$,

$$A\#B = B\#A \quad \text{and} \quad (A\#B) \otimes (B\#A) = (A \otimes B)\#(B \otimes A)$$

and by the operator arithmetic–geometric mean inequality,

$$(A \otimes B)\#(A \otimes B) \leqslant \frac{1}{2}\{(A \otimes B) + (B \otimes A)\}.$$

These imply the inequality (3).

In the case when the Hilbert space is finite dimensional and A and B are diagonal matrices, we can consider the traces of both sides of (3). Taking traces of both sides of (3) gives the discrete Cauchy–Schwarz inequality.

Proposition 8. *Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be non-negative real numbers. Then the following inequality holds*:

$$\left\{\sum_1^n \sqrt{a_i b_i}\right\}^2 \leqslant \sum_1^n a_i \sum_1^n b_i.$$

4.2. Refinement of the Cauchy–Schwarz inequality

Let ξ be a positive scalar valued continuous function on $(0, \infty)$. For positive definite operators A and B, a binary operation σ_ξ is defined by

$$A\sigma_\xi B := A^{\frac{1}{2}}\xi\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)A^{\frac{1}{2}}.$$

Using this, a generalization of Propositions 5 and 4 is described.

Proposition 9. *Let A and B be positive definite operators on a Hilbert space and let f and g be positive scalar valued continuous functions satisfying the three conditions in Proposition 3. If ξ and η are defined by $\xi(t) := f(1, t)$ and $\eta(t) := g(1, t)$, then*

$$
\begin{aligned}
(A\#B)\otimes(A\#B) &\leqslant \frac{1}{2}\{(A\sigma_\xi B)\otimes(A\sigma_\eta B)+(A\sigma_\eta B)\otimes(A\sigma_\xi B)\}\\
&\leqslant \frac{1}{2}\{(A\otimes B)+(B\otimes A)\}.
\end{aligned}
$$

Proof. To show the first inequality, it is sufficient to prove

$$(A\#B)\otimes(A\#B)=\{(A\sigma_\xi B)\otimes(A\sigma_\eta B)\}\#\{(A\sigma_\eta B)\otimes(A\sigma_\xi B)\}$$

since this equality implies the desired result by the arithmetic–geometric mean inequality. The preceding equality is equivalent to the following:

$$\left(C^{\frac{1}{2}}\otimes C^{\frac{1}{2}}\right)=\{(I\sigma_\xi C)\otimes(I\sigma_\eta C)\}\#\{(I\sigma_\eta C)\otimes(I\sigma_\xi C)\},$$

where $C=A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$. From the assumption that f and g satisfy the first condition in Proposition 3, that is $f(a,b)g(a,b)=ab$ $(a,b>0)$, we have $\xi(t)\eta(t)=t$. This equality implies

$$
\begin{aligned}
\{(I\sigma_\xi C)\otimes(I\sigma_\eta C)\}\#\{(I\sigma_\eta C)\otimes(I\sigma_\xi C)\} &= \{\xi(C)\otimes\eta(C)\}\#\{\eta(C)\otimes\xi(C)\}\\
&= \{\xi(C)\#\eta(C)\}\otimes\{\eta(C)\#\xi(C)\}\\
&= \left(C^{\frac{1}{2}}\otimes C^{\frac{1}{2}}\right).
\end{aligned}
$$

To show the second inequality, we put $C=A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$. The above-mentioned equality $\xi(t)\eta(t)=t$ implies that the desired inequality is equivalent to the following one.

$$\{\xi(C)\otimes C\xi(C)^{-1}\}+\{C\xi(C)^{-1}\otimes\xi(C)\}\leqslant(C\otimes 1)+(1\otimes C).$$

Let E be the spectral measure of C (see [4]), that is

$$C=\int_{(0,\infty)} t\,\mathrm{d}E(t).$$

Then the left side of the inequality in question is

$$\int\int_{(0,\infty)^2}\left\{\xi(t)\frac{s}{\xi(s)}+\frac{t}{\xi(t)}\xi(s)\right\}\mathrm{d}E(t)\otimes\mathrm{d}E(s)$$

and the right one is

$$\int\int_{(0,\infty)^2}\{t+s\}\mathrm{d}E(t)\otimes\mathrm{d}E(s).$$

It follows from the third condition in Proposition 3, that is

$$f(1,b)\leqslant f(1,a)\quad\text{and}\quad\frac{f(1,a)}{a}\leqslant\frac{f(1,b)}{b}\quad(b\leqslant a),$$

we have

$$t+s-\left\{\xi(t)\frac{s}{\xi(s)}+\frac{t}{\xi(t)}\xi(s)\right\}=\{\xi(t)-\xi(s)\}\left\{\frac{t}{\xi(t)}-\frac{s}{\xi(s)}\right\}\geqslant 0,$$

which implies that the inequality in question holds. □

When $\xi(t)$ (and hence $\eta(t)$) is operator monotone, the statement of Proposition 9 is generalized to the case of positive semidefinite operators.

Theorem 1. *Let A and B be positive semidefinite operators on a Hilbert space. Then*

$$(A\#B)\otimes(A\#B) \leqslant \frac{1}{2}\left\{(A\sigma B)\otimes(A\sigma^{\perp}B)+(A\sigma^{\perp}B)\otimes(A\sigma B)\right\} \leqslant \frac{1}{2}\{(A\otimes B)+(B\otimes A)\}, \tag{1}$$

where σ and $\sigma^{\perp}$ are an operator mean and its dual in the sense of Kubo and Ando.

Proof. Let ξ and η be the positive operator monotone functions such that $\sigma=\sigma_{\xi}$ and $\sigma^{\perp}=\sigma_{\eta}$. By Proposition 9, for any $\epsilon>0$ the operator inequality (1) in Theorem 1 is valid with A_{ϵ} and B_{ϵ} in place of A and B, respectively. By the property of operator means, as $\epsilon\downarrow 0$, the left side term, the middle term and the right side term converge to

$$(A\#B)\otimes(A\#B),\quad \frac{1}{2}\left\{(A\sigma B)\otimes(A\sigma^{\perp}B)+(A\sigma^{\perp}B)\otimes(A\sigma B)\right\}$$

and

$$\frac{1}{2}\{(A\otimes B)+(B\otimes A)\},$$

respectively. This completes the proof of (1). □

In the case when σ is the α-power mean $\#_{\alpha}$, that is an operator mean whose representing function is t^{α}, the above theorem is just an operator version of Propositions 5 and 6.

Corollary 10. *Let A and B be positive semidefinite operators on a Hilbert space. Then*

$$(A\#B)\otimes(A\#B) \leqslant \frac{1}{2}\{(A\#_{\alpha}B)\otimes(A\#_{1-\alpha}B)+(A\#_{1-\alpha}B)\otimes(A\#_{\alpha}B)\} \leqslant \frac{1}{2}\{(A\otimes B)+(B\otimes A)\}.$$

Moreover, the function

$$\alpha\mapsto(A\#_{\alpha}B)\otimes(A\#_{1-\alpha}B)+(A\#_{1-\alpha}B)\otimes(A\#_{\alpha}B)$$

is monotone decreasing on $[0,1/2]$, *and is monotone increasing on* $[1/2,1]$.

Proof. The first part follows immediately from Theorem 1. For the last part, it suffices to show the monotonicity of the function $t\mapsto a^{t}b^{1-t}+a^{1-t}b^{t}$ for non-negative real numbers a and b, and Corollary 7 show this. □

Before closing this section, we give an application of Theorem 1. We show that the inequality (1) is not only a generalization of the Milne's inequality and the Callebaut inequality, but also a generalization of some inequalities concerning the Schur product of matrices.

It has been known that the Schur product of n-square matrices A and B, that is the entrywise product of A and B, is n-square principal submatrix of the tensor product $A\otimes B$. Thus Theorem 1 implies the following.

Corollary 11. *If A and B are positive semidefinite matrices, then*

$$(A\#B) \circ (A\#B) \leqslant (A\sigma B) \circ (A\sigma^{\perp} B) \leqslant A \circ B,$$

where $\circ$ means the Schur product of A and B.

Remark. In the case that $1 > \lambda > 0$ and $A\sigma B = \lambda A + (1 - \lambda)B$, the above inequality is proved by Ando [1].

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